L'Hospital's Rule Can be Used to Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

Anita Mizer
Dr. Jeffrey Osikiewicz

L'HOSPITAL'S RULE

Suppose f and g are differentiable on (a,b) and $g'(x)\neq 0$ for $a<x<b$. If $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$ and $\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)} = L$.

Abstract

The standard proof presented in calculus courses concludes that $\frac{d}{dx}\sin x = \cos x$ using the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$. A natural question then becomes can you logically use L'Hospital's Rule on this limit? The objective of this project is to name another method to find $\frac{d}{dx}\sin x$ without using the limit $\lim\limits_{x\to 0}\frac{\sin x}{x}$. Once this proof is established, L'Hospital's Rule can then be used on this limit without any logical uncertainties.

1. The Limit

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

SOLUTION.

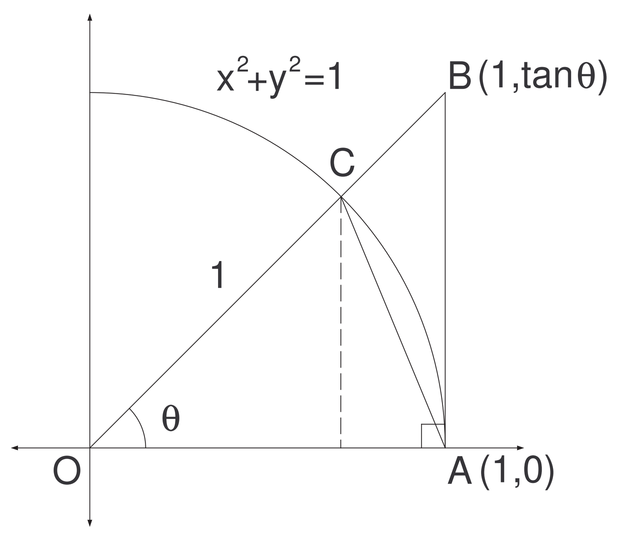

Figure 1

Using Figure 1, we obtain the following equations, which are valid for $0<\theta<\pi/2$:

$$\text{area of triangle } OAC = \frac12\cdot\text{base}\cdot\text{height} = \frac12\cdot 1\cdot\sin\theta = \frac{\sin\theta}{2}$$

$$\text{area of sector } OAC = \frac12\cdot\text{angle}\cdot(\text{radius})^2 = \frac12\cdot\theta\cdot 1^2 = \frac{\theta}{2}$$

$$\text{area of triangle } OAB = \frac12|OA|\,|AB| = \frac12\cdot 1\cdot\tan\theta = \frac12\cdot\frac{\sin\theta}{\cos\theta}.$$

It is geometrically clear that

$$\text{area of triangle } OAC \le \text{area of sector } OAC \le \text{area of triangle } OAB,$$

so that

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac12\cdot\frac{\sin\theta}{\cos\theta}, \qquad \text{for } 0<\theta<\frac{\pi}{2}.$$

If we multiply this inequality by $\frac{2}{\sin\theta}$ and examine the reciprocal of this inequality we obtain

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1, \qquad \text{for } 0<\theta<\frac{\pi}{2}.$$

Since $\lim\limits_{\theta\to 0^+}\cos\theta = 1 = \lim\limits_{\theta\to 0^+} 1$, it follows from the Squeezing Theorem that

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Since $\frac{\sin\theta}{\theta}$ is an even function,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1.$$

Therefore,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

2. Evaluating The Derivative With the Limit

Show that $\frac{d}{dx}\sin x = \cos x$.

SOLUTION.

By definition,

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\sin h-\sin x+\sin x\cos h}{h}\\
&= \lim_{h\to 0}\frac{\cos x\sin h-\sin x\,(1-\cos h)}{h}\\
&= \lim_{h\to 0}\frac{\cos x\sin h}{h} - \lim_{h\to 0}\frac{\sin x(1-\cos h)}{h}\\
&= \cos x\lim_{h\to 0}\frac{\sin h}{h} - \sin x\lim_{h\to 0}\frac{1-\cos h}{h}\\
&= \cos x(1) - \sin x\left[\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)\left(\frac{1+\cos h}{1+\cos h}\right)\right]\\
&= \cos x - \sin x\left[\lim_{h\to 0}\frac{1-\cos^2 h}{h(1+\cos h)}\right]\\
&= \cos x - \sin x\left[\lim_{h\to 0}\frac{\sin^2 h}{h(1+\cos h)}\right]\\
&= \cos x - \sin x\left[\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{1+\cos h}\right]\\
&= \cos x - \sin x\left[(1)\left(\tfrac{0}{1+1}\right)\right]\\
&= \cos x.
\end{aligned}$$

3. Another Method to Evaluate the Derivative

Show that $\frac{d}{dx}\sin x = \cos x$.

SOLUTION.

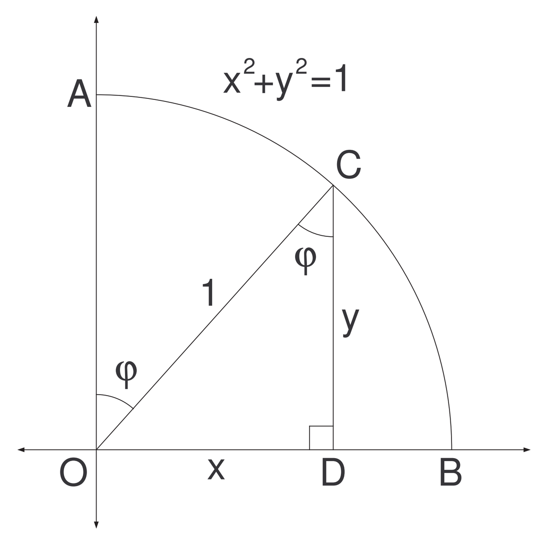

Figure 2

Let ACB in Figure 2 be the first quadrant of the unit circle. The area of $OACD$ is given by

$$\int_0^x\sqrt{1-x^2}\,dx, \qquad 0<x<1$$

This is equivalent to the area of triangle OCD plus area of the sector OAC. So,

$$\begin{aligned}
\int_0^x\sqrt{1-x^2}\,dx &= \frac12 x\sqrt{1-x^2}+\frac12\arcsin x\\
\frac{d}{dx}\left(\int_0^x\sqrt{1-x^2}\,dx\right) &= \frac{d}{dx}\left(\frac12 x\sqrt{1-x^2}\right)+\frac12\frac{d}{dx}\arcsin x\\
\sqrt{1-x^2} &= \frac12\sqrt{1-x^2}+\left(\frac12 x\right)\left[\frac12\left(\frac{-2x}{\sqrt{1-x^2}}\right)\right]+\frac12\frac{d}{dx}\arcsin x\\
\sqrt{1-x^2} &= \frac12\sqrt{1-x^2}-\frac{x^2}{2\sqrt{1-x^2}}+\frac12\frac{d}{dx}\arcsin x\\
\frac{d}{dx}\arcsin x &= 2\left(\sqrt{1-x^2}-\frac12\sqrt{1-x^2}+\frac{x^2}{2\sqrt{1-x^2}}\right)\\
\frac{d}{dx}\arcsin x &= \sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}\\
\frac{d}{dx}\arcsin x &= \frac{1-x^2+x^2}{\sqrt{1-x^2}}\\
\frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } 0<x<1.
\end{aligned}$$

Now we can find the derivative of $\sin x$ by observing that $\sin x$ is the inverse of $\arcsin x$. So, if $\sin x = y(0<x<\frac{\pi}{2}, 0<y<1)$, then $\arcsin y = z$ and, by the Inverse Function Theorem,

$$\begin{aligned}
\frac{d}{dx}\sin x &= \frac{1}{\frac{d}{dy}\arcsin y}\\
&= \frac{1}{\frac{1}{\sqrt{1-y^2}}}\\
&= \sqrt{1-y^2}\\
&= \sqrt{1-\sin^2 x}\\
&= \sqrt{cos^2 x}\\
&= \cos x \qquad \text{for } 0<y<\frac{\pi}{2}.
\end{aligned}$$

Therefore, $\frac{d}{dx}\sin x = \cos x$.(Spiegel, 1956)

We can now evaluate $\lim\limits_{x\to 0^+}\frac{\sin x}{x}$ using l'Hospital's Rule without any uncertainties:

$$\lim_{x\to 0^+}\frac{\sin x}{x} \overset{\text{LH}}{=} \lim_{x\to 0^+}\frac{\cos x}{1} = 1.$$

4. References

M.R. Spiegel, *On the Derivative of Trigonometric Functions*, American Mathematical Monthly, **63** (1956), 118–120.