Choose Ohio First

KENT STATE UNIVERSITY

Error Correction in Finite Arbitrary Length Binary Messages

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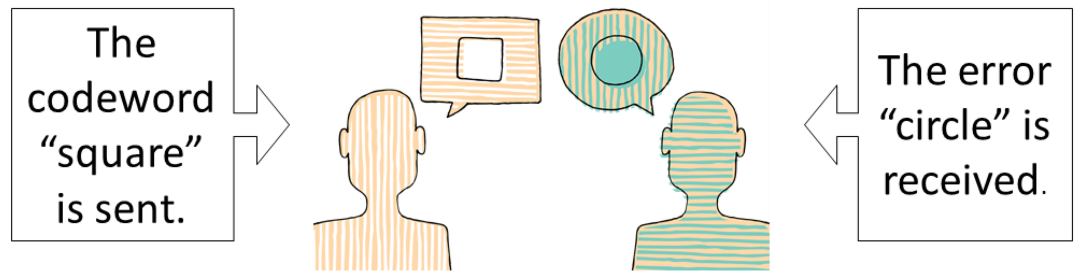
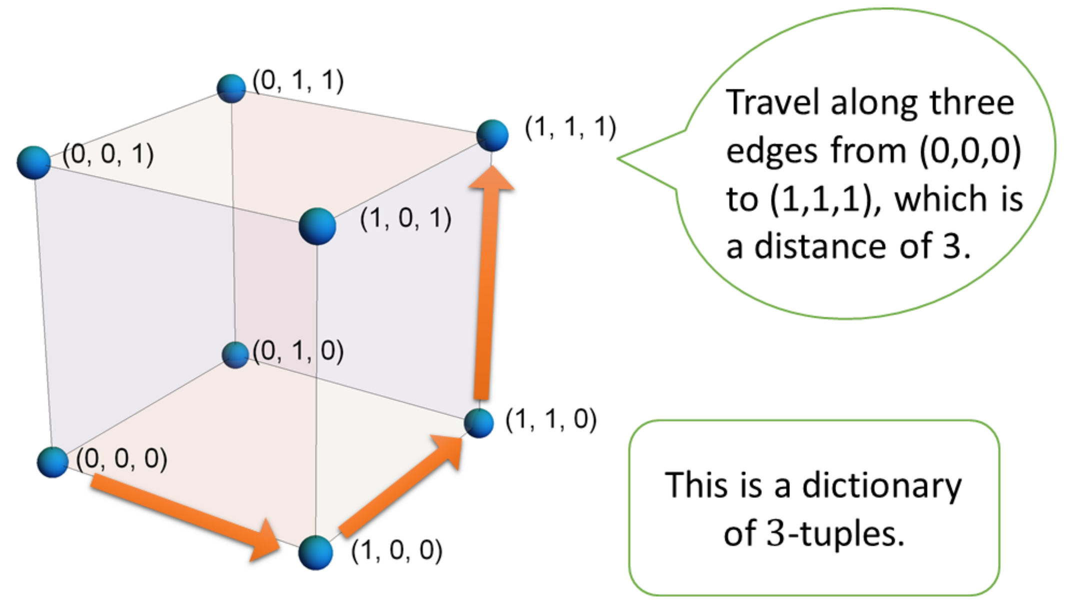
Efficient Digital Communications



1ST Can errors be detected?

2ND Can errors be corrected?

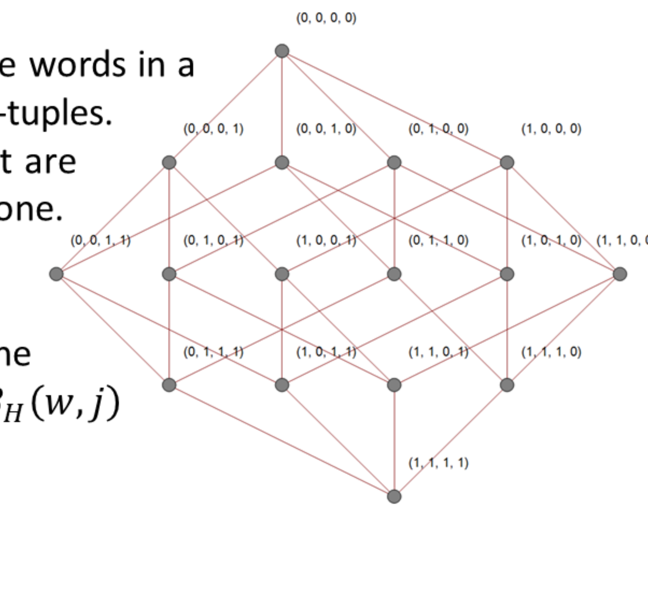
The 3-Dimensional Picture

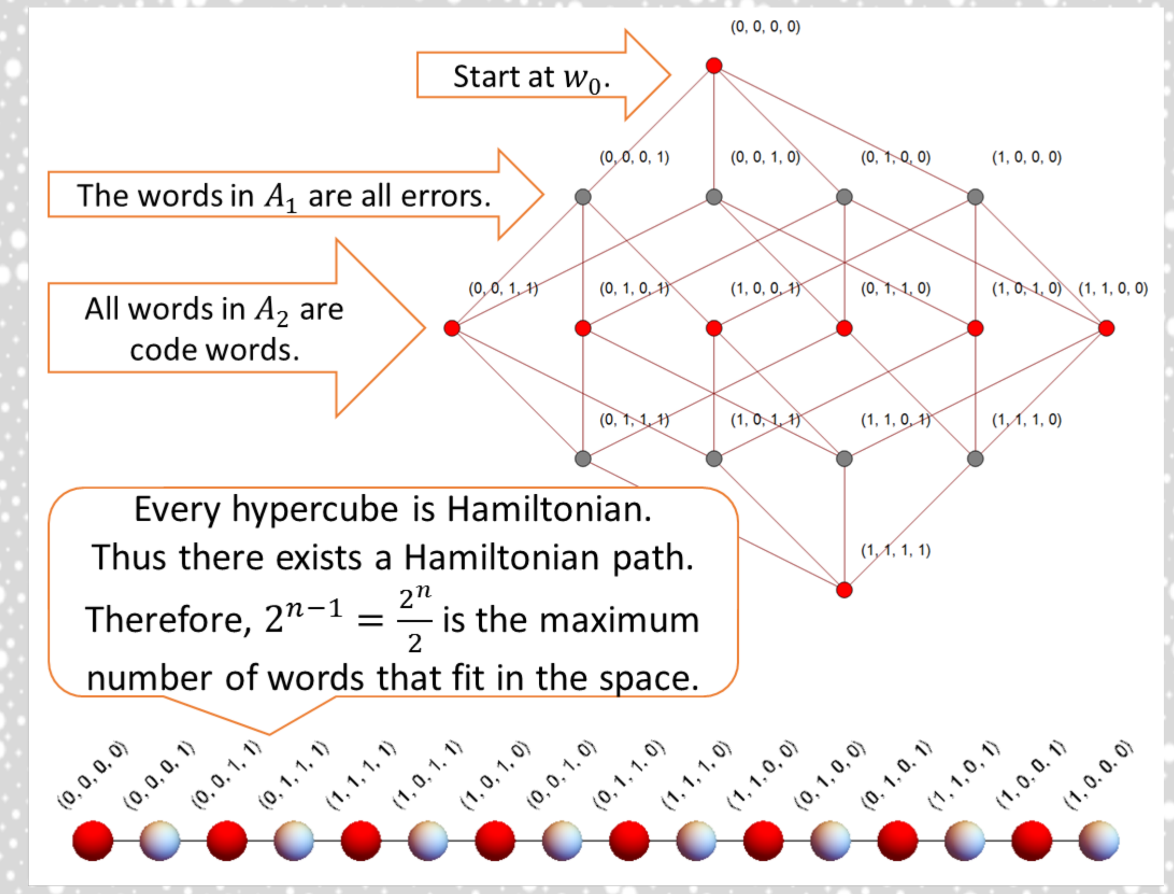


How Many Words Are In a Ball?

- The image shows all of the words in a dictionary consisting of 4-tuples.
- From one level to the next are words with a distance of one.
- The pattern is that of binomial coefficients.
- The general formula for the number of words in the $B_H(w, j)$ where $1 \leq j \leq n$ is

$$\sum_{k=0}^{j} \binom{n}{k}.$$

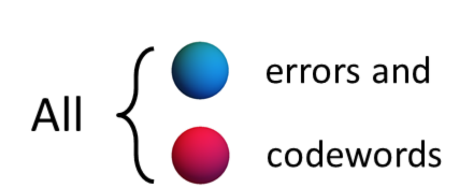




What are the codewords?

- Dictionary: A vector space V over the field $\mathbb{Z}_2$.
- Word: A vector in V. (i.e., a codeword or an error)
- Distance: The number of components that are different between two words.

The Abstract Picture



combined form a vector space V over the finite field $\mathbb{Z}_2$.

Words in an Annulus Have Even Distances

Theorem: Let V be a vector space over $\mathbb{Z}_2$. For any $w_0 \in V$, the set $A_j = \{y: H(w_0, v) = j, \forall v \in V\}$ does not contain any two words an odd distance apart.

Proof. Starting from w_0 words that are a distance of one from w_0, denoted A_1, each have one component that is different from w_0. Each of those words in A_1 has two components different from the other words in A_1.

Assume that the words in A_j all have an even distance from the other words in A_j. Let us consider the set A_{j+1}, and let a be an element of A_{j+1} and let a' be the element in A_j such that one component of a' is transposed to obtain a. By symmetry, we only need to consider the distance between a and the elements of $\{A_{j+1} - a\}$, which we shall denote as B.

Consider the distance between a and some element $b \in B$ such that $b' \in A_j$ and one component of b' is transposed to obtain b. If the component of a' that is transposed is the same component of b' that is transposed, then $H(a, b) = H(a', b')$. Otherwise, $H(a, b) = H(a', b') + 2$. □

Maximum Correctable Words

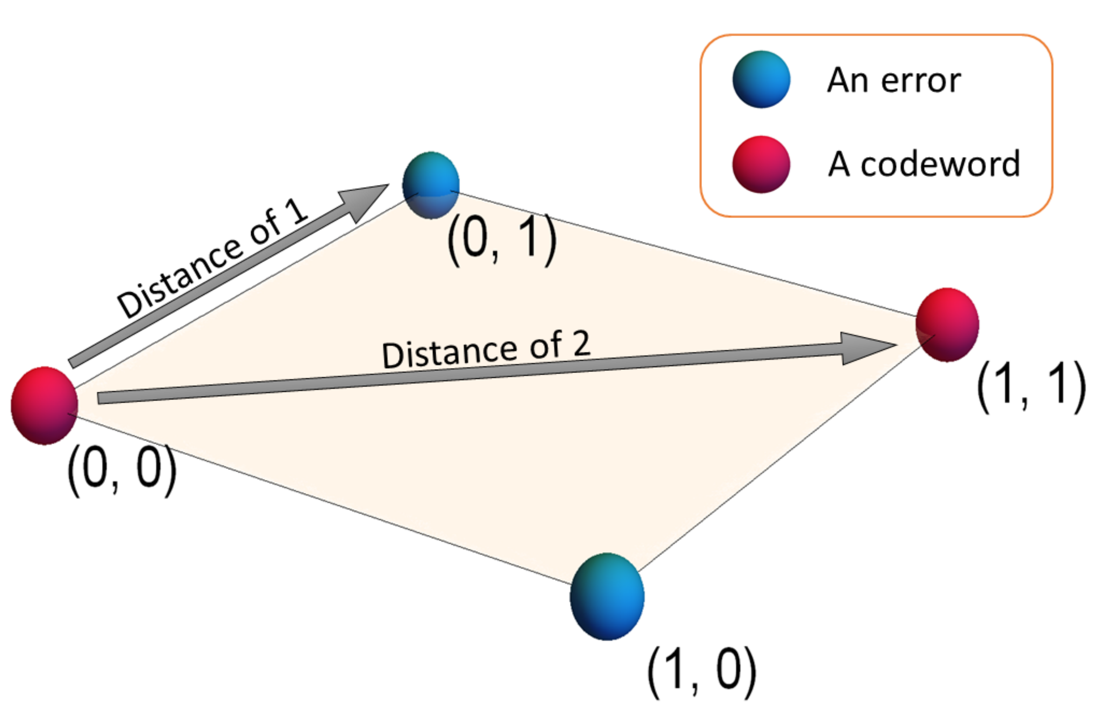
Lower bound: Assuming that the ball of radius 2 centered at any codeword is disjoint for any other similar ball, then the lower bound is:

$$\left\lfloor \frac{2^{n+1}}{n^2 + n + 1} \right\rfloor.$$

Upper bound: Since there is a path connecting all words in the dictionary in a single line, at most every third word can be a codeword:

$$\frac{2^n}{3}$$

A Geometric Interpretation



Symmetry of the Space

Note: We can always let (0,0,…,0) be the first codeword because the space is symmetric.

Proof. Let c' be some codeword in V and $f: V \to V$ be defined as

$$f(c) = \left(\begin{cases} c_1, & c'_1 = 0 \\ c_1 + 1, & c'_1 = 1 \end{cases}, \ldots, \begin{cases} c_n, & c'_n = 0 \\ c_n + 1, & c'_n = 1 \end{cases} \right).$$

It is clear that $f(f(c)) = c$ for any $c \in V$. Thus, f is its own inverse function. A map from one discrete space to another discrete space always maps open sets to open sets. Therefore f is a homeomorphism. □

Maximum Detectable Code Words

- Pick a w_0 to be the first code word.
- No words in the annulus A_1 can be a code word.
- We showed that all words in A_2 are at least a distance of two from each other. Chose all of them as code words.
- Every word in A_3 is a distance of one from some code word in A_2, so there are no code words here.
- Keep choosing every word in every other annulus.
- This gives 2^{n-1} detectable code words.
- But can we fit even more code words into the space?

Conclusions

- The number of single-error correctable codewords in the space $n = 4$ is 2. For $n = 5$, it is 4, and for $n = 6$, we have 8. Therefore, our conjecture is that there are 2^{n-3} codewords in the space.
- Our future research will focus on narrowing the upper and lower bounds of the maximum correctable code words until they converge.
- Additionally, we are investigating the use of generator and parity check matrices in order to

Acknowledgements

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- Colin Adams, and Robert Franzosa. *Introduction to Topology: Pure and Applied*. Upper Saddle River, NJ: Pearson Prentice Hall, 2008. 150-152.
- Sarah Adams, *Introduction to Algebraic Coding Theory*. http://www.math.cornell.edu/~web3360/eccbook2007.pdf, 2008.