The Connection between Ford Circles and Continued Fractions

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Abstract

This research investigates the connections between Ford Circles, Continued Fractions, and Fraction Mediants. Approximating irrational numbers is a longstanding pursuit of mathematics. Part of number theory deals with the approximation of Quadratic Surds, and it was through this lens that I was able to discover a connection between the popular way of approximating Quadratic Surds, Continued Fraction Expansion, and Fraction Mediants. It turns out that the action of taking a mediant between two fractions might be a connected to Continued Fractions.

Definitions:

- **Ford Circle:** Center $= \left({}^{p}/_{q}, \frac{1}{2q^2}\right)$, $\mathbf{Radius} = \frac{1}{2q^2}$
  - where $\frac{p}{q}$ is an irreducible Fraction
  - The circle is tangent to the x-axis
  - And any two Ford Circles are either tangent or disjoint from each other.
- **Mediant:** $\frac{a}{b} \vee \frac{c}{d} = \frac{a+c}{b+d}$

Methods:

Continued Fraction: $[a_0; a_1, a_2, a_3] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3+1}}}$

Convergent $= \dfrac{a_0a_1a_2a_3 + a_0a_1a_2 + a_0a_1 + a_0a_3 + a_0 + a_2a_3 + a_2 + 1}{a_1a_2a_3 + a_1a_2 + a_1 + a_3 + 1}$

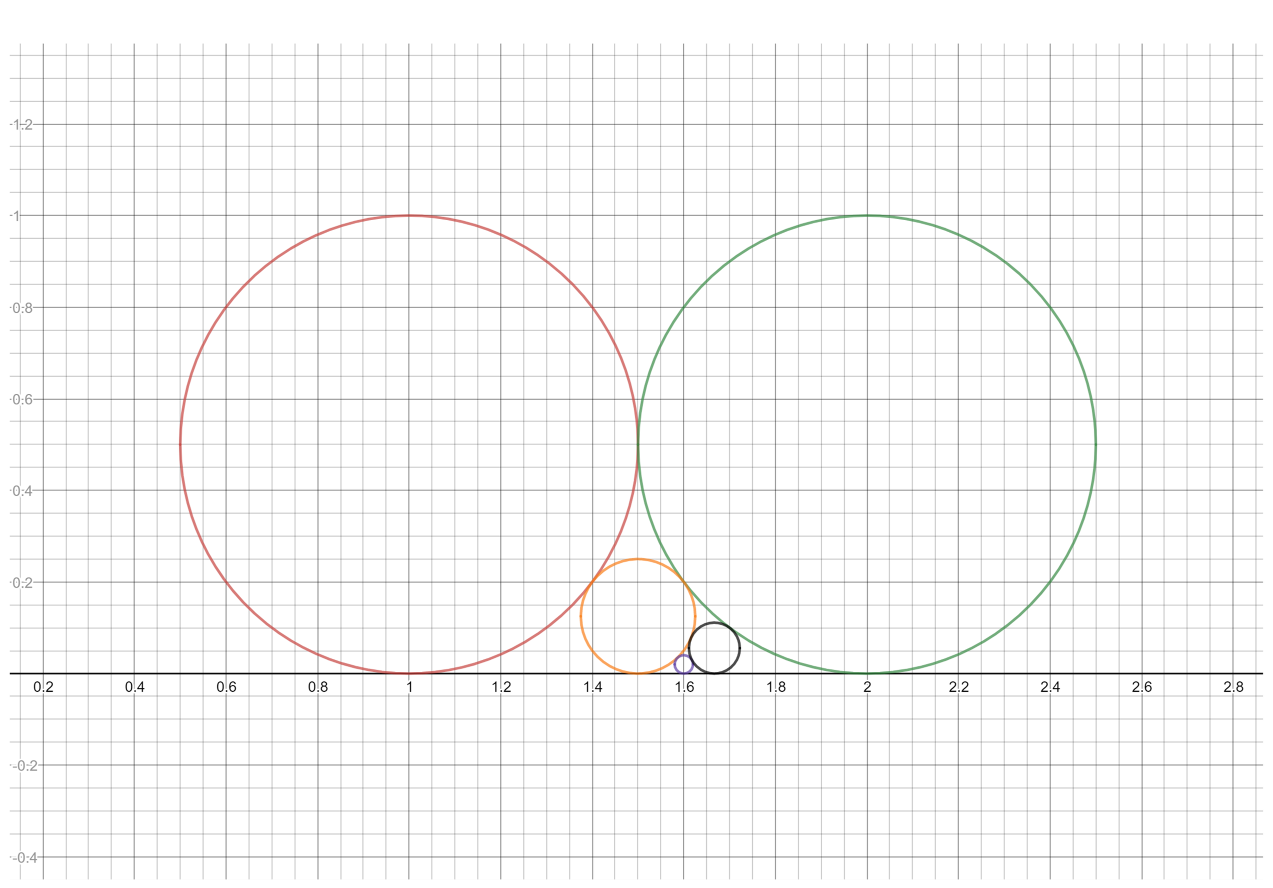
Mediant Convergents:

$$\frac{a_0+1}{1} \vee \frac{a_0}{1}, a_1 \; times = \frac{a_0a_1+a_0+1}{a_1+1}$$

$$\frac{a_0a_1+a_0+1}{a_1+1} \vee \frac{a_0(a_1-1)+a_0+1}{(a_1-1)+1}, a_2 \; times = \frac{a_0a_1a_2+a_2a_1+a_0+a_2+1}{a_1a_2+a_1+1}$$

$$\frac{a_0a_1a_2+a_2a_1+a_0+a_2+1}{a_1a_2+a_1+1} \vee \frac{a_0a_1(a_2-1)+(a_2-1)a_1+a_0+a_2+1}{a_1(a_2-1)+a_1+1}, a_3 \; times = \text{Convergent}$$

Motivation: $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$



Ford Circle Pattern = LR

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

Continued Fraction =[1; 1]

Conjectures:

- Fraction Mediant Operation is well defined between any Continued Fractions
- The Convergents of a Continued Fraction equal the mediant convergents of Ford Circle pattern ending at a switch of L to R

References:

An Illustrated Theory of Numbers by Martin H. Weissman

An Introduction to the Theory of Numbers by G. H. Hardy and E. M. Wright

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